OCMC

Solutions

Ontario Mathematics Competition

2025

1. Compute the value of $1+2+3+4+5+6+7+8+9+10$.

(A) 45 **(B)** 55 **(C)** 61 **(D)** 62 **(E)** 65

proposed by: Daniel Chen

Solution:

$$\begin{aligned} 1+2+3+4+5+6+7+8+9+10 &= \frac{10(11)}{2} \\ &= \boxed{\textbf{(B) } 55}. \end{aligned}$$

2. What is the second smallest four-digit number divisible by 367?

(A) 1057 **(B)** 1101 **(C)** 1196 **(D)** 1278 **(E)** 1468

proposed by: Zheng Wang

Solution: Since $\lceil \frac{1000}{367} \rceil = 3$, the second smallest four-digit number divisible by 367 is $367 \cdot 4 = \boxed{\textbf{(E) } 1468}$.

3. The operator λ is defined so that for any integers a and b, we have $a\lambda b = a^2 + b$. If $(x\lambda 8)\lambda 89 = 2025$, what is the value of x?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 44 **(E)** 1928

proposed by: Daniel Chen

Solution: We have

$$\begin{aligned} (x\lambda 8)\lambda 89 &= 2025 \\ (x^2+8)^2 + 89 &= 2025 \\ (x^2+8)^2 &= 1936. \end{aligned}$$

Since $x^2 + 8 > 0$, we have $x^2 + 8 = 44 \implies x^2 = 36$. Checking the answer choices, $\boxed{\textbf{(A) } 6}$ is the only valid x.

4. There are 90 students in a class. 6 of them like neither apples or bananas, and 15 of them like both apples and bananas. If there is at least one person who only likes bananas, what is the difference between the largest and smallest possible amount of people who do not like bananas?

(A) 48 **(B)** 56 **(C)** 68 **(D)** 76 **(E)** 78

proposed by: Elaine Li

Solution: We see that in the class, $90 - 6 - 15 = 69$ students like exactly one of apples or bananas. The number of people who do not like bananas is the sum of the people who like neither plus the ones who only like apples. The number of people who like neither is constant, so it doesn't matter. The most amount of people who only like apples is 68, since we need to have one person who only likes bananas. The least amount of people who only like apples is 0. Thus, the difference is $68 - 0 =$ $\boxed{\textbf{(C) } 68}$.

5. If the lines $y = 3x + b$ and $y = 3 - 5x$ intersect at a point (p, q) such that $p + q = 11$, find the value of b.

   (A) -3 **(B)** 1 **(C)** 7 **(D)** 19 **(E)** 25

   proposed by: Daniel Chen

   Solution: Since (p, q) is on $y = 3 - 5x$, we know $q = 3 - 5p$. Combined with $p + q = 11$, we find that the desired point is $(-2, 13)$. Thus, $b = q - 3p =$ $\boxed{\textbf{(D) } 19}$.

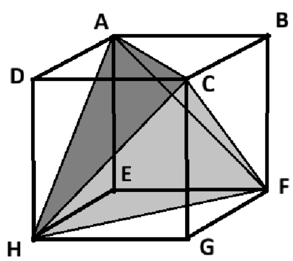
6. Each of the 6 faces of a cube is randomly labelled with a distinct integer from 1 to 6, inclusive. What is the probability that any two opposite faces are labelled with integers that sum to 7?

   (A) $\frac{1}{6}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{15}$ **(D)** $\frac{1}{24}$ **(E)** $\frac{1}{30}$

   proposed by: Leo Wu

   Solution: The probability that 1 is opposite 6 is 1/5. Of the remaining 4 faces, the probability that 2 is opposite 5 is 1/3. After these are locked in place, 3 is forced to be opposite 4. Therefore the answer is $\boxed{\textbf{(C) } 1/15}$.

7. The cube $ABCDEFGH$ has side length 1, as shown in the diagram below. Find the volume of the solid $ACFEH$.



   (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

   proposed by: Elaine Li

   Solution: Note that the volume of the solid $ACFEH$ is equal to the volume of cube $ABCDEFGH$ subtracted by four times the volume of tetrahedron $ADCH$. Since the volume of the cube is $1^3 = 1$ and the volume of a single tetrahedron is $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$, the answer is $1 - 3 \cdot \frac{1}{6} =$ $\boxed{\textbf{(E) } \frac{1}{2}}$.

8. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be seven pairwise distinct positive integers that form a geometric sequence. Given that the number of positive divisors of each of the numbers a_1 and a_7 is 7. What is the total number of positive divisors of a_4?

(A) 7 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 49

proposed by: Terry Yang

Solution: Note that $a_1 = p^6$ and $a_7 = q^6$, where p and q are distinct primes. The common ratio of the considered geometric sequence is $\frac{q}{p}$. Thus, it follows that $a_4 = p^3q^3$. Therefore, the number of divisors of a_4 is $(3+1)(3+1) = \boxed{\textbf{(D) } 16}$.

9. The Fibonacci numbers are defined as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 2$. Let S be the value of the infinite series

$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots$$

where F_n is the n-th Fibonacci number. In which interval does S lie in?

(A) $[2, 3)$ **(B)** $[3, 4)$ **(C)** $[4, 5)$ **(D)** $[5, 6)$ **(E)** $[6, 7)$

proposed by: Christopher Li

Solution: The answer is $\boxed{\textbf{(B) } [3, 4)}$. Firstly,

$$S > \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 3.$$

We have

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_{n-1} + F_{n-2}} \geq 1 + \frac{F_{n-1}}{2F_{n-1}} = \frac{3}{2}.$$

We get

$$\frac{1}{F_n} \leq \frac{F_{n-1}}{F_n} \cdot \frac{F_{n-2}}{F_{n-1}} \cdots \frac{F_2}{F_3} \leq \left(\frac{2}{3}\right)^{n-2}.$$

Thus,

$$S \leq 1 + \left(1 + \frac{2}{3} + \frac{4}{9} + \cdots\right) = 4.$$

But equality cannot occur, so $S \in [3, 4)$.

10. How many permutations (a, b, c, d, e, f, g, h) of $(1, 2, 3, 4, 5, 6, 7, 8)$ satisfy

$$1 < a < b < c > d > e > f < g < h < 8?$$

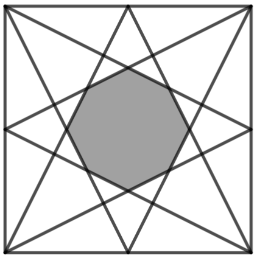
(A) 45 **(B)** 90 **(C)** 120 **(D)** 180 **(E)** 240

proposed by: Leo Wu

Solution: Note that $c = 8$ and $f = 1$ since they cannot fit anywhere else. Once we select the unordered pairs (a, b), (d, e), (g, h) the inequalities uniquely determine the permutation. Therefore the answer is

$$\frac{6!}{(2!)^3} = \boxed{\textbf{(B) } 90}.$$

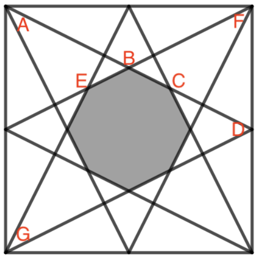
11. In the given figure, the area of the square is 60. By connecting each vertex with the midpoint of the opposite edge, we obtain an octagon. What is the area of this region?



(A) 10 **(B)** $\frac{50}{9}$ **(C)** $6\sqrt{2}$ **(D)** $5 + 3\sqrt{2}$ **(E)** 12

proposed by: Lei He

Solution:



Label points as shown in the above diagram. We wish to find the ratio $x = AB/BC$. Note that from $\triangle GAC \sim \triangle FDC$, we have

$$\frac{CA}{CD} = 2$$
$$\frac{AB + BC}{BD - BC} = 2$$
$$\frac{1 + x}{x - 1} = 2$$
$$x = 3.$$

Since $\triangle ABF \sim \triangle BEC$, we have

$$[BEC] = \frac{1}{9}[ABF] = \frac{1}{9} \cdot \frac{1}{8} \cdot 60 = \frac{5}{6}.$$

We also know $EC = \frac{\sqrt{60}}{3}$. Since the octagon can be divided into a square and 4 copies of $\triangle BEC$, its total area is

$$4 \cdot \frac{5}{6} + \left(\frac{\sqrt{60}}{3}\right)^2 = \boxed{\textbf{(A) } 10}.$$

12. Suppose that $f(x)$ and $g(x)$ are two quadratic polynomials that satisfy the following conditions:

- The first intersection point between $f(x)$ and $g(x)$ is the vertex of $f(x)$ and lies on the y-axis.
- The second intersection point between $f(x)$ and $g(x)$ is the vertex of $g(x)$ and lies on the x-axis.
- The graph of $f(x)$ can be obtained by rotating $g(x)$ about the point $(2, 3)$.

What is the value of $f(1) + g(1)$?

(A) $\frac{27}{8}$ **(B)** $\frac{9}{2}$ **(C)** 6 **(D)** 9 **(E)** $\frac{23}{4}$

proposed by: Daniel Chen

Solution: Let $(0, a)$ be the vertex of the first quadratic and $(b, 0)$ be the vertex of the second quadratic.

Note that the polynomials must be facing different directions, so one opens upwards and the other opens downwards. Therefore, the graph of $f(x)$ is a 180 degree rotation of $g(x)$ around $(2, 3)$. It follows that the midpoint of $(0, a)$ and $(b, 0)$ is $(2, 3)$, so $a = 6$ and $b = 4$.

Consider the polynomial $h(x) = \frac{f(x)+g(x)}{2}$. Since the leading coefficients of $f(x)$ and $g(x)$ are negations of each other, $h(x)$ is a linear polynomial. Also, $h(x)$ passes through $(0, a)$ and $(b, 0)$. We find that $h(x) = -\frac{3}{2}x + 6$. Thus,

$$f(1) + g(1) = 2h(1) = \boxed{\textbf{(D) } 9}.$$

13. Bowl A contains 147 grams of salt dissolved in 4L of water, whereas Bowl B only contains 3L of pure water. Charles first pours 1L of the water in bowl A to B, then pours 1L of the water in bowl B back to A. After repeating this action an infinite amount of times, how much salt is contained in bowl A?

(A) 73.5 **(B)** 84 **(C)** 94.5 **(D)** 105 **(E)** 126

proposed by: Oscar Zhou

Solution: Let the amount of salt at equilibrium (after infinite exchanges) in bowls A and B be a and b, respectively.

After pouring 1L from A to B, there is $\frac{3a}{4}$ and $b + \frac{a}{4}$ grams of salt in bowls A and B, respectively.

After pouring 1L from B to A, now there is $\frac{3a}{4} + \frac{b+\frac{a}{4}}{4}$ and $\frac{3}{4} \cdot (b + \frac{a}{4})$ grams of salt in bowls A and B, respectively.

At equilibrium, the amount of salt in the two bowls do not change after pouring water back and forth.

Thus, $a = \frac{3a}{4} + \frac{b+\frac{a}{4}}{4} = \frac{3a}{4} + \frac{b}{4} + \frac{a}{16}$, meaning that $3a = 4b$.

Since $a + b = 147$, we have $a = \boxed{\textbf{(B) } 84}$.

14. Andy, Benjamin, Carl, David, Edward, and Frank sit in a circle. Andy and Benjamin must sit together, and Carl and Edward cannot sit together. Seating arrangements are considered distinct if one cannot be rotated to match the other. How many distinct seating arrangements are possible?

    (A) 6 **(B)** 20 **(C)** 24 **(D)** 72 **(E)** 120

    proposed by: Jacob Lu

    Solution: We proceed by complementary counting.

    Group Andy and Benjamin as one block (with 2 orders). Then we arrange 5 objects: the AB-block, Carl, David, Edward, Frank. In a circle (rotations equivalent), there are $(5–1)! = 24$ arrangements. Multiplying by 2 (orders of AB) gives 48 total arrangements.

    Now, we find the number of arrangements where Carl and Edward sit together. Treat C and E as a block (with 2 orders) along with the other 3 objects. These 4 objects can be arranged in $(4–1)! = 6$ ways. So the number of arrangements with C and E adjacent is $6 \cdot 2 \cdot 2 = 24$, since we can swap CE and AB.

    Therefore, the number of valid arrangements is $48–24 = \boxed{\textbf{(C) } 24}$.

15. If a, b, and c are real numbers such that one of them is double of another, find the smallest possible value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.

    (A) 2 **(B)** $2+\sqrt{2}$ **(C)** $2\sqrt{3}$ **(D)** 3 **(E)** $\frac{1}{2}+2\sqrt{2}$

    proposed by: Zheng Wang

    Solution: Without loss of generality, there are two possible cases: $a = 2b$ and $b = 2a$.

    If $a = 2b$, we have

$$\begin{aligned}\frac{2b}{b} + \frac{b}{c} + \frac{c}{2b} &= 2 + \frac{b}{c} + \frac{c}{2b} \\ &\geq 2 + \sqrt{2},\end{aligned}$$

    where the last inequality comes from AM-GM. Equality can be achieved through $a = 2$, $b = 1$, $c = \sqrt{2}$.

    If $b = 2a$, we have

$$\begin{aligned}\frac{a}{2a} + \frac{2a}{c} + \frac{c}{a} &= \frac{1}{2} + \frac{2a}{c} + \frac{c}{a} \\ &\geq \frac{1}{2} + 2\sqrt{2},\end{aligned}$$

    where the last inequality comes from AM-GM. Equality can be achieved through $a = 1$, $b = 2$, $c = \sqrt{2}$.

    Since $\frac{1}{2} + 2\sqrt{2} < 2 + \sqrt{2}$, the answer is $\boxed{\textbf{(E) } \frac{1}{2} + 2\sqrt{2}}$.

16. Link repeatedly flips a fair coin until he gets tails, at which point he stops. If the nth flip lands on heads, then he earns n additional rupees. Determine Link's expected total rupees earned from the game.

(A) 1 **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** 2 **(E)** $\frac{5}{2}$

proposed by: Oscar Zhou

Solution: The k-th flip contributes $k(\frac{1}{2})^k$ rupees to the expected earning, so the answer is

$$\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \boxed{\textbf{(D) } 2}.$$

17. A positive integer n is *powerful* if for any integers a, b, and c,

$$n \mid (a+b)(b+c)(c+a)(a-b)(b-c)(c-a).$$

Compute the sum of all powerful integers.

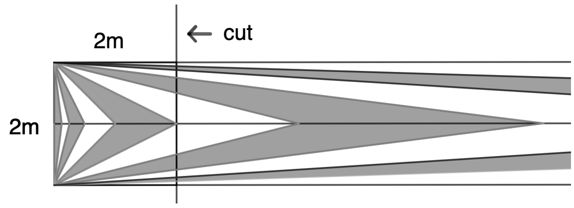
(A) 7 **(B)** 12 **(C)** 28 **(D)** 60 **(E)** 72

proposed by: Leo Wu

Solution: Setting $(a, b, c) = (1, 3, 5), (2, 3, 4)$, all powerful integers must divide $\gcd(3072, 420) = 12$. We now show that 12 always divides the expression.

Rewrite the expression as $(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$. Since x^2 is can only be $0, 1 \pmod 4$, by Pigeonhole, 2 of a^2, b^2, c^2 will cover the same residue and 4 will divide their difference. Similarly, since x^2 can only be $0, 1 \pmod 3$, 2 of a^2, b^2, c^2 will cover the same residue and 3 will divide their difference. Therefore a positive integer is powerful if and only if it divides 12. The answer is $1 + 2 + 3 + 4 + 6 + 12 = \boxed{\textbf{(C) } 28}$.

18. The Aeldari had a legendary black and white flag of 2 meters wide that extendeds infinitely to the right, where each region is twice the length of the previous. A reckless officer of theirs, however, accidentally cut the flag in a single slash of his sword into a square with side length 2 meters. Shame! The original flag is shown below, with the pattern continuing infinitely both near the flagpole and into the vastness of space. What is the area of the shaded region left on the square flag?



(A) $\frac{5}{3}$ **(B)** 2 **(C)** $\frac{23}{12}$ **(D)** $\frac{9}{4}$ **(E)** $\frac{5}{2}$

proposed by: Zheng Wang

Solution: Let the origin be the intersection of the line of symmetry of the flag and the cut. Assume the flag extends to $+\infty$ parallel to the x-axis.

Firstly, consider the triangle region with vertices $(-2,-1)$, $(-2,1)$, and the origin. In this region, every shaded region is to the right of a white region that has half its area. Thus, the shaded region takes up $2/3$ of the total area. The area of the shaded region is $2/3 \cdot 2 = 4/3$.

Now, consider the triangle region with vertices $(0,1)$, $(-2,1)$, and the origin. In this region, every shaded region is above a white region that has double its area. Thus, the shaded region takes up $1/3$ of the total area. The area of the shaded region is $1/3 \cdot 1 = 1/3$. Since this region appears twice in the area left of the cut, the total area of the shaded region is

$$\frac{4}{3} + 2 \cdot \frac{1}{3} = \boxed{\textbf{(B) } 2}.$$

19. There are $2 \leq n \leq 1000$ bowling pins set up in a straight line. Alex and Ben take turns removing k initially consecutive pins, where k is a proper divisor of n. The person who takes the last pin wins. If Alex goes first, find the sum of all integers n for which Ben wins.

    (A) 0 **(B)** 2 **(C)** 38 **(D)** 249999 **(E)** 500499

proposed by: Prince Zhang

Solution: For $n = 2$, Ben wins because Alex takes one pin and Ben takes the remaining pin.

For all $n > 2$, we claim that Alex wins. If n is even, Alex's first move is to take the middle two pins. If n is odd, Alex's first move is to take the middle pin. In both cases, Alex's has split the bowling lines into two equal groups. Alex's winning strategy is to copy whatever move Ben plays in the other group. This way, Alex will remove the last pin.

Thus, the sum of all integers n for which Ben wins is $\boxed{\textbf{(B) } 2}$.

20. Let $p(x)$ represent the number x minus the sum of its digits. Let $p^k(x)$ be $p(p(...p(p(x))...))$, where p is applied to x a total of k times. Given that $5^{10} = 9765625$, what is the value of $p^{2025}(5^{10})$?

    (A) 9635058 **(B)** 9667325 **(C)** 9691254 **(D)** 9711836 **(E)** 9751851

proposed by: Elaine Li

Solution: A number minus the sum of its digits is always divisible by 9. To prove this, write the number as the sum of $a_k 10^k$. We see that $a_k \cdot 10^k - a_k = a_k(10^k - 1)$ and $9 \mid (10^k - 1)$.

Among the answer choices, only A, C and E are divisible by 9. We see the sum of the digits after each $p(x)$ is at most $9 \cdot 7$, so the number is at least $5^{10} - 2025 \cdot 9 \cdot 7 = 9638050$, eliminating option A. Also, the sum of the digits after each $p(x)$ is at least $1 \cdot 7$, so the number is at most $5^{10} - 2025 \cdot 1 \cdot 7 = 9751450$, eliminating option E. Thus, the correct answer is $\boxed{\textbf{(C) } 9691254}$.

21. Let ABC be a triangle such that $AB = 7, BC = 24, CA = 25$. Let D, E be points on the angle bisector of $\angle BAC$ such that BD is parallel to EC. If the midpoint of ED is the incenter

of $\triangle ABC$, what is AD? (the incenter is defined as the intersection of the angle bisectors of $\triangle ABC$)

(A) $\frac{35}{3}$ **(B)** 12 **(C)** $\frac{16\sqrt{3}}{3}$ **(D)** $\frac{15}{2}$ **(E)** $\frac{60}{7}$

proposed by: Christopher Li

Solution: Let F be incenter. Let AD intersect BC at G. Then

$$\frac{EG}{GD} = \frac{CG}{BG} = \frac{AC}{AB} = \frac{25}{7}.$$

The inradius of ABC is 3. We can find $AF = 5$ and $FG = \frac{15}{4}$ by length chasing in the triangle. Let $DG = x$. Then $DF = FE$ implies $EA = x - \frac{5}{4}$. From

$$\frac{25}{7} = \frac{EG}{GD} = \frac{x + \frac{15}{2}}{x},$$

we solve to get $x = \frac{35}{12}$. Thus,

$$AD = x + 5 + \frac{15}{4} = \boxed{\textbf{(A) } \frac{35}{3}}.$$

22. The polynomial $f(x) = x^4 - x^3 + x^2 - x + 100$ has roots r_1, r_2, r_3, r_4. Let the sequence s_n denote $r_1^n + r_2^n + r_3^n + r_4^n$. Find the last 3 digits of $s_9 - s_8 + s_7 - s_6 + s_5 - s_4 + s_3 - s_2 + s_1 - s_0$.

(A) 000 **(B)** 167 **(C)** 343 **(D)** 625 **(E)** 797

proposed by: Elaine Li

Solution: We use the idea that if $x^4 - x^3 + x^2 - x + 100 = 0$ for roots r_1, r_2, r_3, r_4, then

$$\begin{aligned}&(r_1^4 + r_2^4 + r_3^4 + r_4^4) - (r_1^3 + r_2^3 + r_3^3 + r_4^3) + (r_1^2 + r_2^2 + r_3^2 + r_4^2) - (r_1 + r_2 + r_3 + r_4) + 100 \cdot 4 \\ &= s_4 - s_3 + s_2 - s_1 + 100 \cdot 4 \\ &= 0.\end{aligned}$$

Our goal is to transform $f(x)$ into something that looks like $x^9 - x^8 + x^7 + x^6 + \cdots$. Consider the polynomial

$$\begin{aligned}p(x) &= x^5 f(x) - 99x f(x) - 100 f(x) \\ &= (x^9 - x^8 + x^7 - x^6 + 100x^5) - 99(x^5 - x^4 + x^3 - x^2 + 2025x) - 100(x^4 - x^3 + x^2 - x + 100) \\ &= x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 - 100 \cdot 98x - 100^2.\end{aligned}$$

For each root r, $f(r) = 0$, so $p(r) = 0$. Thus,

$$s_9 - s_8 + s_7 - s_6 + s_5 - s_4 + s_3 - s_2 - 9800s_1 - 10000 = 0$$

so we have

$$s_9 - s_8 + s_7 - s_6 + s_5 - s_4 + s_3 - s_2 + s_1 - s_0 = 9800s_1 + 10000 + s_1 - s_0$$

We see that $s_0 = 1 + 1 + 1 + 1 = 4$, and by Vieta's, $s_1 = r_1 + r_2 + r_3 + r_4 = 1$. Thus, the expression is equal to $9800 + 10000 + 1 - 4 = 19801 - 4 = 19797$, and last 3 digits are $\boxed{\textbf{(E) } 797}$.

23. Emma just opened a flower shop and is tracking her hourly sales. She notices an interesting pattern: from one hour to the next starting from hour 1, her sales never drop by more than 1 flower. Suppose that in the 6th hour she sells exactly 4 flowers, how many different possible sales sequences could Emma have had during the first 6 hours?

(A) 210 **(B)** 1638 **(C)** 1820 **(D)** 3640 **(E)** 4368

proposed by: Wendy Xia

Solution: Let a_i denote the sales in each hour, define $b_i = a_i + i$ with $b_i \le b_{i+1}$ and $b_i \ge i$. We know that $b_1 = 1$, $b_6 = 10$. Consider mapping every valid $\{b_i\}$ sequence to a lattice path (consisting of only rights and ups) from $(1,1)$ to $(6,10)$ whose last point on the line $x = i$ is (i, b_i), and vice versa. The problem is then equivalent to counting the lattice paths from $(1,1)$ to $(6,10)$ that never goes below the diagonal from $(2,1)$ to $(6,5)$.

There is a total of $\binom{14}{5}$ paths and we exclude the invalid ones. We claim that there is a bijection between paths going below the $y = x - 1$ diagonal at some point and paths from $(1,1)$ to $(12,4)$.

Proof: Consider an invalid path, let $(x, x-2)$ be the first point where it goes below the diagonal. To reach $(6,10)$ from there requires $(6-x)$ right steps and $(12-x)$ up steps. Reflect the portion of the path after this point across $y = x - 2$. This swaps the number of right and up steps, so the end point becomes $(x + (12-x), x - 2 + (6-x)) = (12,4)$. Conversely, any path to $(12,4)$ necessarily crosses the diagonal, and reflecting it back at the first crossing point recovers the original invalid path. Thus, the mapping is reversible, and we obtain a bijection between invalid paths and paths from the origin to $(12,4)$.

This yields a total of $\binom{16}{5}$ invalid paths. Finally, the answer is $\binom{14}{5} - \binom{14}{3} = \boxed{\textbf{(B) } 1638}$.

24. Let S be the sum of all distinct elements in the set

$$\left\{ \left\lfloor \frac{1^2}{101} \right\rfloor, \left\lfloor \frac{2^2}{101} \right\rfloor, \left\lfloor \frac{3^2}{101} \right\rfloor, \dots, \left\lfloor \frac{101^2}{101} \right\rfloor \right\}.$$

What is the remainder when S is divided by 101?

(A) 1 **(B)** 50 **(C)** 51 **(D)** 69 **(E)** 70

proposed by: Christopher Li

Solution: In the sequence $\frac{1^2}{101}, \frac{2^2}{101}, \dots \frac{50^2}{101}$, the value increases by at most 1. Therefore, the floors are integers 0 to 24. After $\frac{50^2}{101}$, the value increases by at least 1, so all the floors are distinct. We know $\frac{101^2}{101} = 101 \equiv 0 \pmod{101}$, so we just need to find the sum from $\lfloor \frac{51^2}{101} \rfloor$ to

$\lfloor \frac{100^2}{101} \rfloor$. Since $\lfloor x \rfloor = x - \{x\}$, we have

$$\begin{aligned}\sum_{n=51}^{100} \left\lfloor \frac{n^2}{101} \right\rfloor &= \sum_{n=51}^{100} \frac{n^2}{101} - \sum_{n=51}^{100} \left\{ \frac{n^2}{101} \right\} \\ &= \frac{\frac{100\cdot101\cdot201}{6} - \frac{50\cdot51\cdot101}{6}}{101} - \sum_{n=51}^{100} \left\{ \frac{n^2}{101} \right\} \\ &= 2925 - \sum_{n=51}^{100} \left\{ \frac{n^2}{101} \right\}.\end{aligned}$$

Since 101 is prime, there are 50 quadratic residues mod 101, and we know $\{i^2\}_{i=51}^{100}$ covers every quadratic residue. Furthermore, by Fermat's Christmas Theorem, -1 is a quadratic residue, so if a is a quadratic residue, $-a$ is also a quadratic residue. Thus, we can pair up the 50 quadratic residues such that each pair sums to 101. This gives us

$$\sum_{n=51}^{100} \left\{ \frac{n^2}{101} \right\} = \frac{25 \cdot 101}{101} = 25.$$

The final sum is

$$\frac{24 \cdot 25}{2} + 2925 - 25 \equiv \boxed{\textbf{(D) } 69} \pmod{101}.$$

25. Blackbeard the pirate has buried his treasure somewhere in the coordinate plane! There's a $\frac{1}{2}$ chance that the treasure is at $(0,0)$, a $\frac{1}{4}$ chance that it's at $(0,1)$, and in general a 2^{-n-1} chance that it's at $(0,n)$ for any positive integer n. Luffy starts at the point $(-10,0)$ and can only move either up or right by 1 unit on each step. What is the expected number of paths that Luffy can take to reach the treasure?

(A) 167 **(B)** 343 **(C)** 864 **(D)** 975 **(E)** 1024

proposed by: Daniel Chen

Solution: If the treasure is at $(0,n)$, there are $\binom{10+n}{10}$ paths to it. Thus, the expected number of paths is

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{10+n}{10} \frac{1}{2^{n+1}} &= \frac{1}{2} \sum_{n=0}^{\infty} \binom{10+n}{10} \left(\frac{1}{2}\right)^n \\ &= \frac{1}{2} \cdot \frac{1}{(1-\frac{1}{2})^{11}} \\ &= \boxed{\textbf{(E) } 1024}.\end{aligned}$$