OCMC

Solutions

Ontario Invitational Mathematics Exam

2025

1. Find all solutions to the equation $(x+5)(x+7)(x+9)(x+11) = -12$.

   proposed by: Prince Zhang

   Solution: Let $y = x + 8$. Then the equation becomes

$$\begin{aligned}(y-3)(y-1)(y+1)(y+3) &= -12\\(y^2-9)(y^2-1) &= -12\\y^4 - 10y^2 + 21 &= 0.\end{aligned}$$

   Solving the quadratic in terms of y^2, we get $y^2 = 3, 7 \implies y = \pm\sqrt{3}, \pm\sqrt{7}$. Thus,

$$x = y - 8 = \boxed{-8 \pm \sqrt{3}, -8 \pm \sqrt{7}}.$$

2. The infinite blackboard has all the positive integers written on it, in order. Cauchy erases all multiples of 3 on the blackboard, then Gauss erases all multiples of 5 that are left. Finally, Euler writes all multiples of 7 that have been erased back on the blackboard, in their original positions. What is the 634th number left on the blackboard after the procedure?

proposed by: Oscar Zhou

Solution: From 1 to 105, there are 35 multiples of 3, 21 multiples of 5, and 7 multiples of 15. Thus, Cauchy and Gauss erase a total of $35 + 21 - 7 = 49$ numbers out of every 105 numbers.

From 1 to 105, there are 5 multiples of 21, 3 multiples of 45, and 1 multiple of 105. Thus, Euler recovers $5 + 3 - 1 = 7$ numbers out of every 105 numbers.

Therefore, $105 - 49 + 7 = 63$ numbers are kept for every 105 numbers. The 630th number is 1050. The next four numbers are 1051, 1052, 1054, and 1057. Thus, the 634th number is $\boxed{1057}$.

3. Mark the mosquito is escaping from an angry human! He needs to get from the bottom left cube of a $4 \times 5 \times 2$ box to the opposite cube furthest across the space diagonal to escape. Each $1 \times 1 \times 1$ cube of the box is connected to each adjacent cube, and Mark can only make moves to the right, forward, up, and down. If he cannot revisit a cube out of fear of being captured, in how many ways can he escape the 2-cube tall box?

   proposed by: Prince Zhang

   Solution: Mark needs to make a net of 3 moves right, 4 moves forward, and 1 move up. There are $\binom{7}{3}$ ways to pick the path in the xy plane. For each of the first 7 possible xy positions, Mark can choose to switch z levels or not. For the final xy position, there is only one way Mark can finish at the desired cube. Thus, there are $\binom{7}{3} \cdot 2^7 = \boxed{4480}$ paths.

4. Find the number of positive integer pairs (m, n) satisfying both of the conditions:

   (a) $m^2 + n^2 \leq 4050$

   (b) $m^2 n$ divides $n^3 - m^3$.

proposed by: Avneet Prakash

Solution: Let $d = \gcd(m, n)$. let $m = dM$ and $n = dN$. Substituting into the second condition, we have

$$M^2N \mid N^3 - M^3$$
$$N \mid N^3 - M^3$$
$$N \mid M^3.$$

But $\gcd(N, M) = 1$, thus we are forced $N = 1$. Similarly, the divisibility relation gives us $M^2 \mid N^3$, implying $M = 1$. We find that $m = n$. Now, it remains to find the number of positive integers m such that $2m^2 \leq 4050$, which is $\boxed{45}$.

5. Let a_n be a sequence such that $a_1 = 1$ and for all positive integers n,

$$a_{n+1} = (-1)^0 \frac{a_n}{n^2} + (-1)^1 \frac{a_{n-1}}{(n-1)^2} + \cdots + (-1)^{n-1} \frac{a_1}{1}.$$

What is a_{2025}?

proposed by: Elaine Li

Solution: For positive integer $k \geq 2$, we have

$$\begin{aligned}
a_{k+1} &= \frac{a_k}{k^2} + (-1) \cdot \left((-1)^0 \frac{a_{k-1}}{(k-1)^2} + \cdots + (-1)^{k-1} \frac{a_1}{1} \right) \\
a_{k+1} &= \frac{a_k}{k^2} + (-1)a_k \\
a_{k+1} &= a_k \left(\frac{1}{k^2} - 1 \right) \\
a_{k+1} &= -a_k \left(\frac{(k+1)(k-1)}{k^2} \right) \\
\frac{a_{k+1}}{a_k} &= -\frac{(k+1)(k-1)}{k^2}.
\end{aligned}$$

By telescoping, we have

$$\begin{aligned}
\prod_{k=2}^{n-1} \frac{a_{k+1}}{a_k} &= \prod_{k=2}^{n-1} \left(-\frac{(k+1)(k-1)}{k^2} \right) \\
a_n &= (-1)^{n-2} \frac{n}{n-1} \times \frac{1}{2} \\
a_n &= (-1)^n \frac{n}{n-1} \times \frac{1}{2}.
\end{aligned}$$

because $a_2 = 1$. Thus, $a_{2025} = -\frac{2025}{2024 \times 2} = \boxed{-\frac{2025}{4048}}$.

6. Square $ABCD$ of side length 2 has an incircle in it. Point E is on line segment AB and F is on line segment BC, such that EF is tangent to the circle. Find all possible values of the area of $\triangle DEF$.

proposed by: Zheng Wang

Solution: Let G be the midpoint of AB and let H be the midpoint of BC. Let I be the tangency point of the incircle to EF. Let $GE = x$ and $FH = y$. Denote [...] as the area of the polygon with the given vertices.

Note that $[ADCFE]$ is the inradius times the semiperimeter of $ADCFE$. We also know $EF = EI + IF = x + y$. Thus,

$$\begin{aligned}[ADCFE] &= 1 \cdot \frac{2 + 2 + AE + EF + FC}{2} \\ &= \frac{4 + (1 + x) + (x + y) + (1 + y)}{2} \\ &= 3 + x + y.\end{aligned}$$

Finally,

$$\begin{aligned}[DEF] &= [ADCFE] - [ADE] - [DCF] \\ &= 3 + x + y - \frac{2 \cdot AE}{2} - \frac{2 \cdot FC}{2} \\ &= 3 + x + y - (1 + x) - (1 + y) \\ &= \boxed{1}.\end{aligned}$$

7. On a board consisting of 4 squares aligning horizontally and numbered with $0, 1, 2, 3$ in that order, square x has $x^2 + x$ tokens. Two players make moves in alternating turns such that on each move, a player moves one or more tokens from one square to the square to its left. Once all tokens are in square 0, the game ends, and the player who made the last move wins. Harry will move first in one such game. If he and his opponent both play optimally, the move he should begin with is moving x tokens from square y. Find $x \times y$.

proposed by: Lei He

Solution: Call a board *losing* if the person who first plays a move can never win if the second player plays optimally.

Note that the position $(a_0, 0, 0, 0)$ is losing because the other player made the last move. Given a board with numbers (a_0, a_1, a_2, a_3), we claim that the board is losing if and only if $a_1 = a_3$. This is because no matter what the first player does, the second player can play a move to make the board state satisfy $a_1 = a_3$:

(a) If player 1 moves d tokens from square 1 to 0, then player 2 moves d tokens from 3 to 2. Player 2 can do this because $a_3 = a_1 \geq d$.

(b) If player 1 moves d tokens from square 2 to 1, then player 2 moves d tokens from 1 to 0.

(c) If player 1 moves d tokens from square 3 to 2, then player 2 moves d tokens from 1 to 0. Player 2 can do this because $a_1 = a_3 \geq d$.

Eventually, the first player will end up at $(a_0, 0, 0, 0)$, a losing position.

In the problem, the current board state is $(0, 2, 6, 12)$. The only for Harry to force his opponent into a losing position is to move 10 tokens from square 3 to 2. Thus, $x \times y = 10 \times 3 = \boxed{30}$.

8. A subsequence is formed by deleting zero or more elements from a sequence without changing the order of the rest. For example, OME is a subsequence of $OIME$, but OEI is not.

Over all sequences of length 2025 consisting of only the letters U and W, what is the maximum number of subsequences that read UWU?

proposed by: Leo Wu

Solution: The maximum is 675^3, attained by taking 675 Us, then 675 Ws, then another 675 Us.

Define a subsequence as *cute* if it reads UWU and let there be n Ws. If a W has l Us to its left and r Us to its right, there are exactly lr cute subsequences containing this W.

We know $l + r = 2025 - n$, so by AM-GM,

$$lr \leq \frac{(l+r)^2}{4} = \frac{(2025-n)^2}{4}.$$

Summing over all Ws, there are at most $\frac{n(2025-n)^2}{4}$ cute subsequences. By AM-GM,

$$\frac{n(2025-n)^2}{4} = \frac{(2n)(2025-n)(2025-n)}{8} \leq \frac{1}{8}\left(\frac{4050}{3}\right)^3 = 675^3.$$

9. Triangle ABC has $\angle ABC = 60^\circ$. Let O and I be the circumcenter and incenter of $\triangle ABC$, respectively. Show that if O and I are distinct, OI is not perpendicular to BI.

   proposed by: Wendy Xia

   Solution: Let M be the midpoint of minor arc AC on the circumcircle of ABC. Note that $\angle AIC = 90^\circ + \frac{\angle ABC}{2} = 120^\circ$ and $\angle AOC = 2\angle ABC = 120^\circ$. This implies that $AIOC$ is cyclic. By Incenter-Excenter Lemma, the circumcenter of $\triangle AIC$ is M. Thus,

$$\begin{aligned}
\angle BIO &= 180^\circ - \angle OIM \\
&= 180^\circ - \frac{180^\circ - \angle IMO}{2} \\
&= 90^\circ + \angle IMO \\
&> 90^\circ.
\end{aligned}$$

10. Determine all positive integers n for which there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$ such that $a_1, 2a_2, \ldots, na_n$ all leave distinct remainders upon division by n.

proposed by: Leo Wu

Solution: Let d be a divisor of n. Exactly $\frac{n}{d}$ of $\{1, 2, \ldots, n\}$ are multiples of d and exactly $\frac{n}{d}$ of $\{a_1, 2a_2, \ldots, na_n\}$ are multiples of d. If some i not divisible by d satisfies $d \mid a_i$ then $\{da_d, 2da_{(2d)}, \ldots, na_n, ia_i\}$ would be more than $\frac{n}{d}$ multiplies of d, contradiction. Therefore $a_d, a_{2d}, \ldots, a_n$ are precisely the multiples of d. In particular, $a_n = n$.

Suppose that n is not squarefree and let $p^2 \mid n$. Then $pa_p, 2pa_{(2p)}, \ldots, na_n$ are all multiples of p^2, a contradiction since $\frac{n}{p} > \frac{n}{p^2}$.

Suppose that n is divisible by some odd prime p. n is squarefree so $\frac{n}{p}$ is coprime to p, and thus $\frac{n}{p}, 2\frac{n}{p}, \ldots, n$ is a complete residue class mod p. We know that $a_{\frac{n}{p}}, a_{2\frac{n}{p}}, \ldots, a_n$ are the multiples of $\frac{n}{p}$, so they are also a complete residue class mod p. Likewise, $\frac{n}{p}a_{(\frac{n}{p})}, 2\frac{n}{p}a_{(2\frac{n}{p})}, \ldots, na_n$ is a complete residue class mod p.

By Wilson's theorem, the product of a complete residue class, excluding 0, is -1 mod p. Applying this to our three residue classes above, we obtain $(-1)^2 \equiv -1$ mod p, contradiction.

Since n is squarefree and not divisible by any odd prime, $\boxed{n = 1, 2}$. These both work by taking the permutations (1) and $(1, 2)$ respectively.